The Maximum Determinant of an $n \times n$ Lower Hessenberg (0, 1) Matrix

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ABSTRACT

Let $A = (a_{ij})$ be an $n \times n$ (0, 1) matrix which is lower Hessenberg, i.e., $a_{ij} = 0$ for $j > i + 1$. There are 2^{n-1} (possibly nonzero) terms in the determinant of an $n \times n$ lower Hessenberg (0, 1) matrix, so this is a trivial upper bound for the determinant. We define an $n \times n$ (0, 1) matrix D_n and show that this upper bound is

$$\det D_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Here $\det D_n$ is the nth Fibonacci number, i.e.,

$$\det D_n = \det D_{n-1} + \det D_{n-2} \quad \text{and} \quad \det D_1 = \det D_2 = 1.$$

One has $\det D_n \to \infty$ as $n \to \infty$. This answers positively a question due to W. W. Barrett.

1. INTRODUCTION

Let n be a positive integer, $n > 2$. The $n \times n$ (0, 1) matrix $A = (a_{ij})$ is said to be lower Hessenberg if $a_{ij} = 0$ for $j > i + 1$. There are 2^{n-1} (possibly nonzero) terms in the determinant of an $n \times n$ lower Hessenberg (0, 1) matrix. So this is a trivial upper bound for the determinant. In this

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paper we shall find a sharp upper bound for the determinant of a lower Hessenberg (0, 1) matrix.

Define D_n by

$$d_{i,i-k} = \begin{cases} 1, & k \in \{-1,0,2,4,\dots \mid i-k>0\}, \\ 0 & \text{otherwise}. \end{cases}$$

For example,

$$D_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $\det D_1 = \det D_2 = 1$, and it is not difficult to see that $\det D_n = \det D_{n-1} + \det D_{n-2}$. Thus $\det D_n$ is the nth Fibonacci number,

$$\det D_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

2. RESULT

Define $n \times n$ permutation matrices P_1 and P_2 as follows: P_1 is obtained from the identity matrix by interchanging the last two rows, and P_2 by interchanging the first two columns.

THEOREM. *Let A_n be any $n \times n$ lower Hessenberg* (0, 1) *matrix*, $n > 2$. *Then*

$$|\det A_n| \leqslant \det D_n.$$

Furthermore, $\det A_n = \det D_n$ *if and only if* $A_n = D_n$ *or* $P_1 A_n P_2 = D_n$, *and* $\det A_n = -\det D_n$ *if and only if* $P_1 A_n = D_n$ *or* $A_n P_2 = D_n$.

Proof. To establish the inequality $|\det A_n| \leqslant \det D_n$ it suffices to show that $\det A_n \leqslant \det D_n$ for any $n \times n$ lower Hessenberg (0, 1) matrix A_n. For then, since $P_1 A_n$ is also a lower Hessenberg (0, 1) matrix, $\det(P_1 A_n) \leqslant \det D_n$, which is equivalent to $\det A_n \geqslant -\det D_n$. Similarly, the cases of equality ($\det A_n = -\det D_n$) follow from those for $\det A_n = \det D_n$.

Note that $\det A_2 \leqslant 1$ for any 2×2 (0, 1) matrix A_2. The proof is by induction on n.

If $n = 3$,

$$\det A_3 = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$\leqslant 2 = \det D_3.$$

Also $\det A_3 = 2$ if and only if $a_{11} = a_{22} = a_{33} = a_{12} = a_{23} = a_{31} = 1$, $a_{21} = a_{32} = 0$, so

$$A_3 = D_3.$$

If $n = 4$,

$$\det A_4 = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} & 0 \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & 0 \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}.$$

If either $a_{11} = 0$ or $a_{12} = 0$, then $\det A_4 \leqslant \det D_3 < \det D_4$ by the result for $n = 3$. So assume $a_{11} = a_{12} = 1$. If $a_{23} = 0$,

$$\det A_4 = \begin{vmatrix} 1 & 1 \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \leqslant 1 < \det D_4.$$

So assume $a_{23} = 1$.

Suppose now that $a_{21} = 0$. Then

$$\det A_4 = \begin{vmatrix} a_{22} & 1 & 0 \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$$

$$\leqslant \det D_3 + 1 = \det D_4,$$

and equality holds if and only if

$$\begin{vmatrix} a_{22} & 1 & 0 \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = D_3 \quad \text{and} \quad \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} = 1,$$

that is, if $A_4 = D_4$.

Now consider the case $a_{21} = 1$. Then

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

If $a_{22} = 1$,

$$\det A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$\leqslant \det D_3 < \det D_4.$$

Finally, assume $a_{22} = 0$. Then

$$P_1 A_4 P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ a_{42} & a_{41} & a_{43} & a_{44} \\ a_{32} & a_{31} & a_{33} & a_{34} \end{bmatrix}.$$

By the same argument as used above,

$$\det A_4 = \det P_1 A_4 P_2 \leqslant \det D_3 + 1 = \det D_4,$$

and equality occurs if and only if $P_1 A_4 P_2 = D_4$.

Now we hypothesize $\det A_{n-2} \leqslant \det D_{n-2}$, with equality if and only if $A_{n-2} = D_{n-2}$ or $P_1 A_{n-2} P_2 = D_{n-2}$, and $\det A_{n-1} \leqslant \det D_{n-1}$, with equality if and only if $A_{n-1} = D_{n-1}$ or $P_1 A_{n-1} P_2 = D_{n-1}$. Then

$$\det A_n = \begin{vmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ \hdotsfor{5} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & 0 \\ a_{32} & a_{33} & \cdots & 0 \\ \hdotsfor{4} \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & 0 \\ a_{31} & a_{33} & \cdots & 0 \\ \hdotsfor{4} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

If either $a_{11} = 0$ or $a_{12} = 0$. Then $\det A_n \leqslant \det D_{n-1} < \det D_n$ by the result for $n - 1$. So assume $a_{11} = a_{12} = 1$. if $a_{23} = 0$,

$$\det A_n = \begin{vmatrix} 1 & 1 \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & 0 & \cdots & 0 \\ a_{43} & a_{44} & a_{45} & \cdots & 0 \\ \hdotsfor{5} \\ a_{n3} & a_{n4} & a_{n5} & \cdots & a_{nn} \end{vmatrix}$$

$$\leqslant \det D_{n-2} < \det D_n.$$

So assume $a_{23} = 1$.

Suppose now that $a_{21} = 0$. Then

$$\det A_n = \begin{vmatrix} a_{22} & 1 & 0 & \cdots & 0 \\ a_{32} & a_{33} & a_{34} & \cdots & 0 \\ \hdotsfor{5} \\ a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{34} & 0 & \cdots & 0 \\ a_{41} & a_{44} & a_{45} & \cdots & 0 \\ \hdotsfor{5} \\ a_{n1} & a_{n4} & a_{n5} & \cdots & a_{nn} \end{vmatrix}$$

$$= \det B_{n-1} + \det B_{n-2}$$

$$\leqslant \det D_{n-1} + \det D_{n-2} = \det D_n,$$

and equality holds if and only if

$$B_{n-1} = D_{n-1} \text{ or } P_1 D_{n-1} P_2, \quad \text{and} \quad B_{n-2} = D_{n-2} \text{ or } P_1 D_{n-2} P_2.$$

There are four possibilities:

Case 1: $B_{n-1} = P_1 D_{n-1} P_2$ *and* $B_{n-2} = D_{n-2}$. The first equation implies that $a_{n,n-1} = 1$, and the second implies that $a_{n,n-1} = 0$, which is impossible.

Case 2: $B_{n-1} = D_{n-1}$ *and* $B_{n-2} = P_1 D_{n-2} P_2$. Similarly these equations imply a conflicting value for $a_{n,n-1}$.

Case 3: $B_{n-1} = P_1 D_{n-1} P_2$ *and* $B_{n-2} = P_1 D_{n-2} P_2$. If n is an even number, the first equation implies that $a_{n4} = 0$, and the second that $a_{n4} = 1$, which is impossible. Similarly, if n is an odd number, it is also impossible.

Case 4: $B_{n-1} = D_{n-1}$ *and* $B_{n-2} = D_{n-2}$. It follows that $A_n = D_n$.

It remains to consider the case $a_{21} = 1$. Then

$$A_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & a_{22} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

If $a_{22} = 1$,

$$\det A_n = -\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{34} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$$

$$\leqslant \det D_{n-1} < \det D_n.$$

Finally, assume $a_{22} = 0$. Then

$$P_1 A_n P_2 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ a_{32} & a_{31} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n1} & a_{n3} & \cdots & a_{nn} \\ a_{n-1,2} & a_{n-1,1} & a_{n-1,3} & \cdots & a_{n-1,n} \end{bmatrix}.$$

By the same argument as used above,

$$\det A_n = \det P_1 A_n P_2 \leqslant \det D_{n-1} + \det D_{n-2} = \det D_n,$$

and equality occurs if and only if $P_1 A_n P_2 = D_n$. ∎

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Received 20 August 1991; final manuscript accepted 25 March 1992